

On the hadronic contribution to sterile neutrino production

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ABSTRACT: Sterile neutrinos with masses in the keV range are considered to be a viable candidate for warm dark matter. The rate of their production through active-sterile neutrino transitions peaks, however, at temperatures of the order of the QCD scale, which makes it difficult to estimate their relic abundance quantitatively, even if the mass of the sterile neutrino and its mixing angle were known. We derive here a relation, valid to all orders in the strong coupling constant, which expresses the production rate in terms of the spectral function associated with active neutrinos. The latter can in turn be expressed as a certain convolution of the spectral functions related to various mesonic current-current correlation functions, which are being actively studied in other physics contexts. In the naive weak coupling limit, the appropriate Boltzmann equations can be derived from our general formulae.

KEYWORDS: Neutrino Physics, Cosmology of Theories beyond the SM, Thermal Field Theory, Nonperturbative Effects.

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1. Introduction

The problem of explaining the nature of Dark Matter is a central one for cosmology. The most popular attempt is to postulate the existence of a relatively heavy Cold Dark Matter (CDM) particle related, perhaps, to softly broken supersymmetry. However, other particle physics candidates can also be envisaged, and there has been a recent revival particularly in Warm Dark Matter (WDM) scenarios.

While WDM is just an alternative to CDM from the Dark Matter point of view, the issue becomes quite intriguing when other physics considerations are added to the picture. In particular, WDM could be realised through the existence of right-handed sterile neutrinos [1] (see also ref. [2]). This possibility may then lead to some astrophysical applications [3]. Moreover, if there are three sterile neutrinos in total, one for each known generation, then they can be combined to a minimal theoretical framework, dubbed the ν MSM in ref. [4], and used to explain also the known properties of neutrino oscillations [4] and baryogenesis [5, 6]. A phenomenologically successful implementation can be obtained provided all three right-handed neutrinos have masses smaller than the electroweak scale. The role of WDM is played by the lightest right-handed neutrino, whereas the two other

ones should have masses in the range $\mathcal{O}(1-20)$ GeV and be very degenerate to produce the observed baryon asymmetry [6]. In addition, an extension of the ν MSM by a real scalar field, inflaton, allows to incorporate inflation [7].

Apart from its mass, M_s , the WDM neutrino is also characterised by its mixing angle with active neutrinos, θ . At present the strongest observational constraints on M_s and θ come from two sides: structure formation in the form of Lyman- α forest observations imposes a stringent lower limit on M_s [8, 9], while X-ray constraints exclude the region of having simultaneously a “large” M_s and θ [10]–[14]. More precisely, if the average momentum of sterile neutrinos at temperatures of a few MeV coincides with that of active neutrinos then, according to ref. [9], the WDM neutrino cannot be lighter than $M_s \simeq 14$ keV. For masses in this range the mixing angle cannot exceed $\theta \simeq 2.9 \times 10^{-3} (1 \text{ keV}/M_s)^{5/2}$ [13].

In this corner of the parameter space the interactions of sterile neutrinos with the particles of the Minimal Standard Model (MSM) are so weak that the sterile neutrinos cannot equilibrate in the Early Universe via active-sterile transitions [1]. Therefore, information about initial conditions does *not* get lost; initial conditions do in general play a role for the current abundance. Evidently, it would be convenient to fix the initial conditions at some temperature low enough such that only the ν MSM degrees of freedom play a role. Apart from initial conditions, one also has to fix the values of all nearly conserved quantum numbers in the MSM, such as the lepton and baryon asymmetries. Alternatively, one can specify the physics beyond the ν MSM and thus determine the initial conditions dynamically; an example of such a computation can be found in ref. [7], where the main source of sterile neutrino production was associated with inflation.

One can imagine circumstances, however, where the initial conditions only play a subdominant role. In this case the production mechanism of the sterile neutrino WDM can be attributed to active-sterile neutrino mixing, as in the Dodelson-Widrow scenario [1]. The requirements for this situation can be formulated as follows: suppose that the initial condition is that the concentration of sterile neutrinos is zero (which is possible if the interactions of sterile neutrinos with all particles beyond the ν MSM such as the inflaton are negligible) and that there are no significant lepton asymmetries (lepton asymmetries corresponding to a chemical potential $\mu/T \gtrsim 10^{-3}$ would play an important role in sterile neutrino production [15]). Suppose also that the two heavier sterile neutrinos, present in the ν MSM, are heavy enough so that their decays in the Early Universe do not produce any entropy [16]. Then the WDM abundance can be expressed as a function of the mass M_s and the angle θ [1, 10, 17, 18]. Matching the observed abundance one gets a relation between M_s and θ , which can be confronted with the observational bounds mentioned above.

Approximate M_s - θ relations derived along these lines have been presented in Refs. [1, 10, 17, 18]. According to the most recent analysis [18],

$$\theta \approx 1.3 \times 10^{-4} \left(\frac{1 \text{ keV}}{M_s} \right)^{0.8}, \tag{1.1}$$

where a dark matter abundance $\Omega_{\text{DM}} \approx 0.22$ and a QCD crossover transition temperature

$T_{\text{QCD}} \approx 170$ MeV have been inserted. If true, the combination of the Lyman- α bounds [9] and X-ray bounds [13] mentioned above rules out the Dodelson-Widrow scenario. However, other production mechanism such as resonant production due to large lepton asymmetries [15] (see also ref. [19]) or due to inflaton interactions [7] are feasible [16].

Due to the importance of the problem our aim here is to reanalyse the M_s - θ relation within the Dodelson-Widrow scenario. In fact, a computation of the sterile neutrino production rate represents a very non-trivial theoretical challenge. The reason is that the region of temperatures at which sterile neutrinos are produced most intensely is [1]

$$T \sim 150 \text{ MeV} \left(\frac{M_s}{1 \text{ keV}} \right)^{\frac{1}{3}}. \tag{1.2}$$

At higher temperatures their production is suppressed because of medium effects [20]. The temperature in eq. (1.2) is very close to the pseudocritical temperature of the QCD crossover. Therefore, neither the dilute hadronic gas approximation nor the weakly interacting quark-gluon plasma picture is expected to provide for an accurate description.

The presence of strongly interacting hadrons at these temperatures leads to two sources of uncertainties. A well-known one is related to the hadronic equation of state, needed for the time-temperature relation in the expanding Universe, entering the sterile neutrino production equation. Unfortunately, experiments with heavy ion collisions cannot directly measure the equation of state of hadronic matter. In addition, present lattice simulations with light dynamical quarks involve uncontrolled systematic uncertainties, such as the absence of a continuum limit extrapolation. In other words, the equation of state of QCD is only known approximately in this temperature range and contains significant systematic uncertainties (for a recent discussion, see ref. [21]).

At the same time, the equation of state does matter in the computation of the sterile neutrino abundance. As has been mentioned already in ref. [1] and elucidated further in Refs. [17], even the purely leptonic contribution to the abundance depends significantly on the effective number of massless degrees of freedom, g_* , which in turn changes dramatically, from about 60 down to about 20, when the quark-gluon plasma cools to a hadronic gas. However, neither the uncertainties of the equation of state, nor the subsequent uncertainties in the M_s - θ relation, have been exhaustively investigated in these works.

There is also a second type of a hadronic uncertainty, which has a dynamical character. Sterile neutrinos can be produced in reactions of two types, the first containing leptons only and the second having hadrons in the initial state. The hadronic reactions were omitted in refs. [1, 10]. Processes with quarks were mentioned in ref. [17] (not in ref. [18]), but without an explanation of how they were treated in the QCD crossover region.

It is this second uncertainty that is the focus of the present paper. Our goal is to set up a general formalism for attacking it. In a later work, a numerical analysis of the M_s - θ relation and its uncertainties will be presented. More concretely, the current number density of the WDM neutrinos in the Dodelson-Widrow scenario is given by $\theta^2 F + \mathcal{O}(\theta^4)$. In this paper we derive an expression which allows to relate the coefficient F to a certain

equilibrium Green's function, the so-called active neutrino spectral function, defined within the MSM, while all dependence on the parameters of the ν MSM appears as the prefactor θ^2 .

It is appropriate to stress that analogous relations exist in other contexts as well. For instance, it can be shown that the photon spectral function computed within the MSM determines the production rate of on-shell photons and dilepton pairs from strongly interacting systems such as colliding heavy nuclei [22], and the production rate of active neutrino pairs from hot/dense astrophysical environments such as the cores of neutron stars [23]. Nevertheless, we are not aware of the existence of such relations in the present context, so we want to discuss their derivation in a hopefully pedagogic manner.

Given the Green's function, it should still be evaluated. As we have already mentioned, this turns out to be a very difficult task, since the sterile neutrino production rate peaks at temperatures of the order of the QCD scale. In this temperature range strong interactions play a dominant role, and perturbative methods fail. In the second part of our paper, we thus show how the active neutrino spectral function can be related to various vector and axial-vector current-current correlation functions defined within high temperature QCD. Such objects have previously been studied with a variety of methods, such as chiral perturbation theory, QCD sum rules, lattice QCD, and resummed weak-coupling perturbation theory, and also possess independent physics applications, particularly in connection with the photon and dilepton pair production rate computations mentioned above.

The plan of this paper is the following. In section 2 we derive the expression alluded to above, expressing the sterile neutrino production rate in terms of the spectral function of active neutrinos, computable within the MSM. In section 3 we relate the hadronic contribution to the active neutrino spectral function to mesonic current-current correlation functions, which can be defined within QCD. We also show how the result reduces to certain Boltzmann equations in the naive (unresummed) weak-coupling limit. We conclude and outline future prospects in section 4. The three appendices contain certain basic definitions for the various bosonic and fermionic Green's functions that appear in our study, and an alternative derivation for section 2.

2. General formula for the sterile neutrino production rate

2.1 Notation

It is a matter of convention whether the right-handed neutrinos are represented as Weyl, right-handed Dirac, or Majorana fermions. Choosing here the last option, the Minkowskian Lagrangian of ν MSM can be written as

$$\mathcal{L} = \frac{1}{2} \bar{\tilde{N}}_I i \not{\partial} \tilde{N}_I - \frac{1}{2} M_I \bar{\tilde{N}}_I \tilde{N}_I - F_{\alpha I} \bar{L}_\alpha \tilde{\phi} a_R \tilde{N}_I - F_{\alpha I}^* \tilde{N}_I \tilde{\phi}^\dagger a_L L_\alpha + \mathcal{L}_{\text{MSM}}, \quad (2.1)$$

where \tilde{N}_I are Majorana spinors, repeated indices are summed over, M_I are Majorana masses that we have chosen to be real in this basis, L_α are the weak interaction eigenstates of the active lepton doublets, $F_{\alpha I}$ are elements of a 3×3 complex Yukawa matrix, $\tilde{\phi} = i\tau_2 \phi^*$ is the conjugate Higgs doublet, and $a_L \equiv (1 - \gamma_5)/2$, $a_R \equiv (1 + \gamma_5)/2$ are chiral projectors.

After electroweak symmetry breaking, $\langle \tilde{\phi} \rangle = (v/\sqrt{2}, 0)$, we define the matrix

$$(M_D)_{\alpha I} \equiv \frac{v F_{\alpha I}}{\sqrt{2}}, \tag{2.2}$$

where $v \simeq 246$ GeV. The various neutrino fields, active and sterile, then couple to each other, and diagonalising the mass matrix we can define mass eigenstates in the usual way. However, a sterile neutrino of type I couples to an active neutrino mass eigenstate of type a with a very small angle only,

$$\theta_{Ia} \simeq \sum_{\alpha=1}^3 \frac{(M_D^\dagger)_{I\alpha}}{M_I} U_{\alpha a}, \tag{2.3}$$

where $U_{\alpha a}$ is the mixing matrix between the active neutrino interaction and mass bases. As mentioned above, the phenomenologically relevant part of the parameter space corresponds to values $\theta \ll 1$. Therefore, the sterile neutrino interaction eigenstate of type I is to a very good approximation also a sterile neutrino mass eigenstate, and we can ignore the distinction in the following. To fix the conventions, $I = 1$ corresponds to the lightest sterile neutrino, contributing to warm dark matter.

The Green's functions that we will need involve a lot of sign and other conventions whose definitions are unfortunately not unique in the literature. We therefore explicitly state our conventions in Appendices A and B, for Green's functions made out of bosonic and fermionic operators, respectively. Our metric convention is $(+---)$.

2.2 Derivation of the master equation

According to our assumption, the concentration of sterile neutrinos was zero at very high temperatures, $T \gg 1$ GeV. Moreover, because of the smallness of its Yukawa coupling, the lightest sterile neutrino never equilibrated. In this section we show that these two facts allow us to express the production rate of this neutrino through a certain well-defined *equilibrium* Green's function within the MSM. The consideration below is very general and uses only the basic principles of thermodynamics and quantum field theory. In particular, it does not require any solution of kinetic equations, nor a discussion of coherence or its loss due to collisions, or the like.

The general way we proceed with the derivation is equivalent to how fluctuation-dissipation relations, or linear response formulae, are usually derived (see, e.g., Refs. [24, 25]). There exists, however, also an alternative derivation, which makes more direct contact with particle states and the related transition matrix elements and which is also somewhat shorter. The price is that this derivation appears to be slightly less rigorous. Nevertheless, the end result is identical, so we present the alternative derivation in Appendix C.

We disregard first the Universe expansion, which can be added later on (cf. eq. (2.22)). Let $\hat{\rho}$ be the density matrix for ν MSM, incorporating all degrees of freedom, and \hat{H} the corresponding full Hamiltonian operator. Then the equation for the density matrix is

$$i \frac{d\hat{\rho}(t)}{dt} = [\hat{H}, \hat{\rho}(t)]. \tag{2.4}$$

We now split \hat{H} in the form

$$\hat{H} = \hat{H}_{\text{MSM}} + \hat{H}_{\text{S}} + \hat{H}_{\text{int}}, \quad (2.5)$$

where \hat{H}_{MSM} is the complete Hamiltonian of the MSM, \hat{H}_{S} is the free Hamiltonian of sterile neutrinos, and \hat{H}_{int} , which is proportional to the sterile neutrino Yukawa couplings, contains the interactions between sterile neutrinos and the particles of the MSM. To find the concentration of sterile neutrinos, one has to solve eq. (2.4) with some initial condition. Following [1], we will *assume* that the initial concentration of sterile neutrinos is zero, that is

$$\hat{\rho}(0) = \hat{\rho}_{\text{MSM}} \otimes |0\rangle\langle 0|, \quad (2.6)$$

where $\hat{\rho}_{\text{MSM}} = Z_{\text{MSM}}^{-1} \exp(-\beta \hat{H}_{\text{MSM}})$, $\beta \equiv 1/T$, is the equilibrium MSM density matrix at a temperature T , and $|0\rangle$ is the vacuum state for sterile neutrinos. The physical meaning of eq. (2.6) is clear: it describes a system with no sterile neutrinos, while all MSM particles are in thermal equilibrium.

Considering now $\hat{H}_0 = \hat{H}_{\text{MSM}} + \hat{H}_{\text{S}}$ as a “free” Hamiltonian, and \hat{H}_{int} as an interaction term, one can derive an equation for the density matrix in the interaction picture, $\hat{\rho}_{\text{I}} \equiv \exp(i\hat{H}_0 t) \hat{\rho} \exp(-i\hat{H}_0 t)$, in the standard way:

$$i \frac{d\hat{\rho}_{\text{I}}(t)}{dt} = [\hat{H}_{\text{I}}(t), \hat{\rho}_{\text{I}}(t)]. \quad (2.7)$$

Here, as usual, $\hat{H}_{\text{I}} = \exp(i\hat{H}_0 t) \hat{H}_{\text{int}} \exp(-i\hat{H}_0 t)$ is the interaction Hamiltonian in the interaction picture. Now, perturbation theory with respect to \hat{H}_{I} can be used to compute the time evolution of $\hat{\rho}_{\text{I}}$; the first two terms read

$$\hat{\rho}_{\text{I}}(t) = \hat{\rho}_0 - i \int_0^t dt' [\hat{H}_{\text{I}}(t'), \hat{\rho}_0] + (-i)^2 \int_0^t dt' \int_0^{t'} dt'' [\hat{H}_{\text{I}}(t'), [\hat{H}_{\text{I}}(t''), \hat{\rho}_0]] + \dots, \quad (2.8)$$

where $\hat{\rho}_0 \equiv \hat{\rho}(0) = \hat{\rho}_{\text{I}}(0)$. Note that perturbation theory with \hat{H}_{I} breaks down at a certain time $t \simeq t_{\text{eq}}$ due to so-called secular terms. After t_{eq} sterile neutrinos enter thermal equilibrium and their concentration needs to be computed by other means. For us $t \ll t_{\text{eq}}$ and perturbation theory works well.

We are interested in the distribution function of the sterile neutrinos. It is associated with the operator

$$\frac{d\hat{N}_I}{d^3\mathbf{x} d^3\mathbf{q}} \equiv \frac{1}{V} \sum_{s=\pm 1} \hat{a}_{I;\mathbf{q},s}^\dagger \hat{a}_{I;\mathbf{q},s}, \quad (2.9)$$

where $\hat{a}_{I;\mathbf{q},s}^\dagger$ is the creation operator of a sterile neutrino of type I , momentum \mathbf{q} , and spin state s , normalised as

$$\{\hat{a}_{I;\mathbf{p},s}, \hat{a}_{J;\mathbf{q},t}^\dagger\} = \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta_{IJ} \delta_{st}, \quad (2.10)$$

and V is the volume of the system. Then the distribution function $dN_I/d^3\mathbf{x} d^3\mathbf{q}$ (number of sterile neutrinos of type I per $d^3\mathbf{x} d^3\mathbf{q}$) is given by

$$\frac{dN_I(x, \mathbf{q})}{d^3\mathbf{x} d^3\mathbf{q}} = \text{Tr} \left[\frac{d\hat{N}_I}{d^3\mathbf{x} d^3\mathbf{q}} \hat{\rho}_{\text{I}}(t) \right]. \quad (2.11)$$

One can easily see that the first term in eq. (2.8) does not contribute in eq. (2.11) since \hat{H}_I is linear in $\hat{a}_{I;\mathbf{q},s}^\dagger$ and $\hat{a}_{I;\mathbf{q},s}$. Thus, we get that to $\mathcal{O}(\theta^2)$ the rate of sterile neutrino production reads

$$\frac{dN_I(x, \mathbf{q})}{d^4x d^3\mathbf{q}} = -\frac{1}{V} \text{Tr} \left\{ \sum_{s=\pm 1} \hat{a}_{I;\mathbf{q},s}^\dagger \hat{a}_{I;\mathbf{q},s} \int_0^t dt' [\hat{H}_I(t), [\hat{H}_I(t'), \hat{\rho}_0]] \right\}. \quad (2.12)$$

For small temperatures $T \ll M_W$, the Higgs field in \hat{H}_I can safely be replaced through its vacuum expectation value, so that eqs. (2.1), (2.2) imply

$$\hat{H}_I = \int d^3\mathbf{x} \left[(M_D)_{\alpha I} \hat{\nu}_\alpha a_R \hat{N}_I + (M_D^*)_{\alpha I} \hat{N}_I a_L \hat{\nu}_\alpha \right], \quad (2.13)$$

where now \hat{N}_I is a Majorana spinor field operator. The \hat{N}_I can be treated as free on-shell field operators and can hence be written as

$$\hat{N}_I(x) = \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{s=\pm 1} \left[\hat{a}_{I;\mathbf{p},s} u(I; \mathbf{p}, s) e^{-iP \cdot x} + \hat{a}_{I;\mathbf{p},s}^\dagger v(I; \mathbf{p}, s) e^{iP \cdot x} \right], \quad (2.14)$$

$$\hat{N}_I(x) = \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{s=\pm 1} \left[\hat{a}_{I;\mathbf{p},s}^\dagger \bar{u}(I; \mathbf{p}, s) e^{iP \cdot x} + \hat{a}_{I;\mathbf{p},s} \bar{v}(I; \mathbf{p}, s) e^{-iP \cdot x} \right], \quad (2.15)$$

where we assumed the normalization in eq. (2.10), and $p^0 \equiv E_{\mathbf{p}}^{(I)} \equiv \sqrt{\mathbf{p}^2 + M_I^2}$, $P \equiv (p^0, \mathbf{p})$. The spinors u, v satisfy the completeness relations $\sum_s u(I; \mathbf{p}, s) \bar{u}(I; \mathbf{p}, s) = \not{p} + M_I$, $\sum_s v(I; \mathbf{p}, s) \bar{v}(I; \mathbf{p}, s) = \not{p} - M_I$, and their Majorana character requires that $u = C \bar{v}^T$, $v = C \bar{u}^T$, where C is the charge conjugation matrix. Inserting the free field operators into eq. (2.13), we can rewrite it as

$$\hat{H}_I = \int d^3\mathbf{x} \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{s=\pm 1} \left[\hat{a}_{I;\mathbf{p},s}^\dagger \hat{J}_{I;\mathbf{p},s}(x) e^{iP \cdot x} + \hat{J}_{I;\mathbf{p},s}^\dagger(x) \hat{a}_{I;\mathbf{p},s} e^{-iP \cdot x} \right], \quad (2.16)$$

where

$$\hat{J}_{I;\mathbf{p},s}(x) \equiv -(M_D)_{\alpha I} \hat{\nu}_\alpha(x) a_R v(I; \mathbf{p}, s) + (M_D^*)_{\alpha I} \bar{u}(I; \mathbf{p}, s) a_L \hat{\nu}_\alpha(x). \quad (2.17)$$

It remains to take the following steps:

- (i) We insert eq. (2.16) into eq. (2.12) and remove the sterile neutrino creation and annihilation operators, by making use of eq. (2.10).
- (ii) This leaves us with various types of two-point correlators of the active neutrino field operators. We now note that correlators of the type $\langle \hat{\nu}_\beta(x') \hat{\nu}_\alpha(x) \rangle$ and $\langle \hat{\nu}_\beta(x') \hat{\nu}_\alpha(x) \rangle$, where $\langle \dots \rangle \equiv \text{Tr}[\hat{\rho}_{\text{MSM}}(\dots)]$ and we have generalised the notation so that α, β incorporate also the Dirac indices, vanish, since lepton numbers are conserved within the MSM.
- (iii) The non-vanishing two-point functions contain the spinors u, v in a form where the standard completeness relations mentioned above can be used. The mass terms M_I that are induced this way get projected out by a_L, a_R .

(iv) Introducing the notation in eqs. (B.1), (B.2), the remaining two-point correlator can be written as

$$\langle \hat{\nu}_\alpha(x') \hat{\nu}_\beta(x) + \hat{\nu}_\beta(x') \hat{\nu}_\alpha(x) \rangle = \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot (x-x')} \left[\Pi_{\alpha\beta}^>(-P) - \Pi_{\alpha\beta}^<(P) \right]. \quad (2.18)$$

There is another term with the same structure but with $x \leftrightarrow x'$.

(v) It remains to carry out the integrals over the space and time coordinates. Taking the limit $t \rightarrow \infty$, they yield

$$\lim_{t \rightarrow \infty} \int d^3 \mathbf{x} \int d^3 \mathbf{x}' \int_0^t dt' \left[e^{i(Q-P) \cdot (x-x')} + e^{i(P-Q) \cdot (x-x')} \right] = V (2\pi)^4 \delta^{(4)}(P-Q), \quad (2.19)$$

which allows to cancel $1/V$ from eq. (2.12) and remove P -integration from eq. (2.18).

As a result of all these steps, we obtain

$$\frac{dN_I(x, \mathbf{q})}{d^4 x d^3 \mathbf{q}} = \frac{1}{(2\pi)^3 2q^0} (M_D^*)_{\alpha I} (M_D)_{\beta I} \text{Tr} \left\{ \mathcal{Q} a_L \left[\Pi_{\alpha\beta}^>(-Q) - \Pi_{\alpha\beta}^<(Q) \right] a_R \right\}, \quad (2.20)$$

where we have returned to the convention that α, β label generations, and have expressed the Dirac part through a trace. Inserting eq. (B.8); making use of the fact that $1 - n_F(-q^0) = n_F(q^0)$, where $n_F(x) \equiv 1/[\exp(\beta x) + 1]$; and observing that lepton generation conservation within the MSM restricts the indices α, β to be equal, we finally obtain the master relation

$$\frac{dN_I(x, \mathbf{q})}{d^4 x d^3 \mathbf{q}} = R(T, \mathbf{q}) \equiv \frac{2n_F(q^0)}{(2\pi)^3 2q^0} \sum_{\alpha=1}^3 |M_D|_{\alpha I}^2 \text{Tr} \left\{ \mathcal{Q} a_L \left[\rho_{\alpha\alpha}(-Q) + \rho_{\alpha\alpha}(Q) \right] a_R \right\}, \quad (2.21)$$

where ρ is called the spectral function (eq. (B.3)). We stress again that this relation is valid only provided that the number density of sterile neutrinos created is much smaller than their equilibrium concentration, which however is always the case in the phenomenologically interesting part of the parameter space, at least for $I = 1$.

In an expanding Universe, with a Hubble rate H , the physical momenta redshift as $\mathbf{q}(t) = \mathbf{q}(t_0) a(t_0)/a(t)$, where $a(t)$ is the scale factor. This implies that the time derivative gets replaced with $d/dt = \partial/\partial t - H q_i \partial/\partial q_i$ [26], and eq. (2.21) becomes

$$\left[\frac{\partial}{\partial t} - H q_i \frac{\partial}{\partial q_i} \right] \frac{dN_I(x, \mathbf{q})}{d^3 \mathbf{x} d^3 \mathbf{q}} = R(T, \mathbf{q}). \quad (2.22)$$

3. Hadronic contribution to the active neutrino spectral function

3.1 Notation

As stated by eq. (2.21), we need to estimate the active neutrino spectral function within the MSM. Given that higher-order corrections can be important, this task has to be consistently formulated within finite-temperature field theory. There are in principle two ways to go

forward, the real-time and the imaginary-time formalisms. We follow here the latter since it can be set up also beyond perturbation theory.

Within the imaginary-time formalism, the spectral function can be obtained through a certain analytic continuation of the Euclidean active neutrino propagator. With the conventions specified in Appendix B, we denote the Euclidean propagator by $\Pi_{\alpha\beta}^E(\tilde{q}_0, \mathbf{q})$ (cf. eq. (B.7)). Carrying out an analytic continuation, we define

$$\Pi_{\alpha\beta}^E(-i[q^0 \pm i0^+], \mathbf{q}) \equiv \text{Re } \Pi_{\alpha\beta}^R(q^0, \mathbf{q}) \pm i \text{Im } \Pi_{\alpha\beta}^R(q^0, \mathbf{q}), \quad (3.1)$$

where $\Pi_{\alpha\beta}^R$ is called the retarded Green's function (cf. eq. (B.4)). The relation shown in eq. (3.1) follows from the spectral representations in eqs. (B.9), (B.11). Note that the imaginary part in eq. (3.1) is *defined* as the discontinuity across the real axis:

$$\text{Im } \Pi_{\alpha\beta}^R(q^0, \mathbf{q}) \equiv \frac{1}{2i} \text{Disc } \Pi_{\alpha\beta}^E(-iq^0, \mathbf{q}) \quad (3.2)$$

$$\equiv \frac{1}{2i} \left[\Pi_{\alpha\beta}^E(-i[q^0 + i0^+], \mathbf{q}) - \Pi_{\alpha\beta}^E(-i[q^0 - i0^+], \mathbf{q}) \right] \quad (3.3)$$

$$= \rho_{\alpha\beta}(q^0, \mathbf{q}), \quad (3.4)$$

where the last step introduced the spectral function (eq. (B.3)) and made use of eq. (B.12). As the imaginary-time neutrino propagator is time-ordered by construction (cf. eq. (B.7)), we can use functional integrals for its determination, whereby operator labels can be dropped from the fields from now on.

In order to compute the Euclidean propagator, from which the spectral function follows through eqs. (3.1), (3.4), we need to define the Euclidean theory. Given that we are interested in low temperatures, we can work within the Fermi-model. The interactions of the active neutrinos with the hadronic degrees of freedom on which we concentrate in this paper, are then contained in the Lagrangian

$$\mathcal{L}_E = 2\sqrt{2}G_F \left(\bar{\nu}_\alpha \tilde{\gamma}_\mu a_L l_\alpha \tilde{H}_\mu^W + \tilde{H}_\mu^{W\dagger} \bar{l}_\alpha \tilde{\gamma}_\mu a_L \nu_\alpha + \frac{1}{2} \bar{\nu}_\alpha \tilde{\gamma}_\mu a_L \nu_\alpha \tilde{H}_\mu^Z \right), \quad (3.5)$$

$$\tilde{H}_\mu^W = \bar{d}'_{\beta B} \tilde{\gamma}_\mu a_L u_{\beta B}, \quad \tilde{H}_\mu^{W\dagger} = \bar{u}_{\beta B} \tilde{\gamma}_\mu a_L d'_{\beta B}, \quad (3.6)$$

$$\tilde{H}_\mu^Z = \bar{u}_{\beta B} \tilde{\gamma}_\mu \left(\frac{1}{2} - \frac{4x_W}{3} - \frac{\gamma_5}{2} \right) u_{\beta B} + \bar{d}'_{\beta B} \tilde{\gamma}_\mu \left(-\frac{1}{2} + \frac{2x_W}{3} + \frac{\gamma_5}{2} \right) d_{\beta B}, \quad (3.7)$$

where α, β are generation indices, B is a colour index, $x_W \equiv \sin^2 \theta_W$, and the Fermi-constant reads $G_F = g_w^2/4\sqrt{2}m_W^2$. All fermions are Dirac fields. The fields $d'_{\beta B}$ are related to the mass eigenstates $d_{\beta B}$ with the usual CKM matrix. The Euclidean γ -matrices are defined by $\tilde{\gamma}_0 \equiv \gamma^0$, $\tilde{\gamma}_i \equiv -i\gamma^i$, and satisfy $\tilde{\gamma}_\mu^\dagger = \tilde{\gamma}_\mu$, $\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_{\mu\nu}$. We have defined $\gamma_5 = \tilde{\gamma}_0 \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. Repeated μ -indices are summed over, and that they are both down reminds us of the fact that we are in the Euclidean space-time. We also denote $\tilde{\mathcal{Q}} \equiv \tilde{q}_\mu \tilde{\gamma}_\mu$ and note that if we carry out the Wick-rotation $\tilde{q}_0 \rightarrow -iq^0$ (cf. eq. (3.1)) and simultaneously decide to express the result in terms of Minkowskian rather than Euclidean Dirac-matrices, then

$$i\tilde{\mathcal{Q}} \rightarrow \mathcal{Q} \equiv q_\mu \gamma^\mu. \quad (3.8)$$

3.2 General structure of the active neutrino spectral function

Suppose now that we compute the full Euclidean neutrino self-energy within the MSM. Since only left-handed neutrinos experience interactions in the MSM, we expect the corresponding Euclidean action to have the structure

$$S_E = \int_{\tilde{Q}_f} \sum_{\alpha=1}^3 \bar{\nu}_\alpha(\tilde{Q}) a_R [i\tilde{\not{Q}} + i\tilde{\not{Y}}_{\alpha\alpha}(\tilde{Q})] a_L \nu_\alpha(\tilde{Q}), \quad (3.9)$$

where we have defined the Fourier transforms as $\nu_\alpha(\tilde{x}) = \int_{\tilde{Q}} \exp(i\tilde{Q} \cdot \tilde{x}) \nu_\alpha(\tilde{Q})$, $\bar{\nu}_\alpha(\tilde{x}) = \int_{\tilde{Q}} \exp(-i\tilde{Q} \cdot \tilde{x}) \bar{\nu}_\alpha(\tilde{Q})$. To keep the future expressions more compact, we have also factored the chiral projectors outside of $\tilde{\Sigma}$, but their existence should be kept in mind in the following. With the conventions of eq. (B.7), this leads to the Euclidean propagator

$$\Pi_{\alpha\alpha}^E(\tilde{Q}) = a_L \frac{1}{-i\tilde{\not{Q}} + i\tilde{\not{Y}}(-\tilde{Q})} a_R = a_L \frac{i\tilde{\not{Q}} + i\tilde{\not{Y}}(\tilde{Q})}{[\tilde{Q} + \tilde{\Sigma}(\tilde{Q})]^2} a_R, \quad (3.10)$$

where we have made use of the property $\tilde{\not{Y}}(-\tilde{Q}) = -\tilde{\not{Y}}(\tilde{Q})$, following from hermiticity (or, to be more precise, the so-called γ_5 -hermiticity that replaces hermiticity in the Euclidean theory: the Dirac operator D satisfies $\gamma_5 D^\dagger \gamma_5 = D$) and CP-invariance. We have also left out the flavour indices from $\tilde{\Sigma}$ to compactify the notation somewhat.

Defining now, in analogy with the Wick-rotation of \tilde{Q} , a four-vector $\Sigma_\mu \equiv (i\tilde{\Sigma}_0, \tilde{\Sigma}_i)$; recalling that $\tilde{Q}^2 = \tilde{q}_\mu \tilde{q}_\mu = -Q^2$; and making use of eq. (3.8), we can write a general analytic continuation of $\Pi_{\alpha\alpha}^E(\tilde{Q})$ as

$$\Pi_{\alpha\alpha}^R(q^0, \mathbf{q}) = \Pi_{\alpha\alpha}^E(-iq^0, \mathbf{q}) = a_L \frac{-\not{Q} - \not{Y}(Q)}{Q^2 + 2Q \cdot \Sigma(Q) + \Sigma^2(Q)} a_R. \quad (3.11)$$

Writing finally $\Sigma(q^0 \pm i0^+, \mathbf{q}) \equiv \text{Re } \Sigma(q^0, \mathbf{q}) \pm i \text{Im } \Sigma(q^0, \mathbf{q})$ in analogy with eq. (3.1), making use of eq. (3.4), and employing the symmetry properties $\text{Re } \Sigma(-Q) = -\text{Re } \Sigma(Q)$, $\text{Im } \Sigma(-Q) = \text{Im } \Sigma(Q)$, the master relation of eq. (2.21) becomes

$$\frac{dN_I(x, \mathbf{q})}{d^4x d^3\mathbf{q}} = \frac{4n_F(q^0)}{(2\pi)^3 2q^0} \sum_{\alpha=1}^3 \frac{|M_D|_{\alpha I}^2}{\{[Q + \text{Re } \Sigma]^2 - [\text{Im } \Sigma]^2\}^2 + 4\{[Q + \text{Re } \Sigma] \cdot \text{Im } \Sigma\}^2} \times \quad (3.12)$$

$$\times \text{Tr} \left\{ \not{Q} a_L \left(2[Q + \text{Re } \Sigma] \cdot \text{Im } \Sigma [\not{Q} + \text{Re } \not{Y}] - \{[Q + \text{Re } \Sigma]^2 - [\text{Im } \Sigma]^2\} \text{Im } \not{Y} \right) a_R \right\},$$

where $\Sigma \equiv \Sigma_{\alpha\alpha}(Q)$, and $Q^2 = M_I^2$.

We remark that the trace over Dirac matrices on the latter row of eq. (3.12) could trivially be carried out. Given that this does not simplify the structure in an essential way, however, we do not write down the corresponding formula explicitly. We also note that \not{Y} can contain two types of Lorentz structures, $\not{Y}(Q) = \not{Q} f_1(Q^2, Q \cdot u) + \not{u} f_2(Q^2, Q \cdot u)$, where $u = (1, \mathbf{0})$ is the plasma four-velocity [27]. However, we do not need to make a distinction between these two structures here.

Now, in the absence of (leptonic) chemical potentials [15], it is easy to see that $\text{Re } \Sigma$ gets no contributions at $\mathcal{O}(g_w^2/m_W^2)$, corresponding to 1-loop level within the Fermi model. The dominant contributions are $\mathcal{O}(g_w^2/m_W^4)$, and originate from 1-loop graphs within the electroweak theory [28, 29]. The dominant contributions within the Fermi-model are of 2-loop order, $\mathcal{O}(g_w^4/m_W^4)$, and thus suppressed with respect to the 1-loop effects from the electroweak theory. In contrast, $\text{Im } \Sigma$ requires on-shell particles on the inner lines, and cannot at low energies $E \ll m_W$ get generated within 1-loop level in the electroweak theory (more precisely, $\text{Im } \Sigma$ is exponentially suppressed by $\sim \exp(-m_W/T)$). The dominant contributions are $\mathcal{O}(g_w^4/m_W^4)$ and can be computed within the Fermi-model. It is these contributions that we concentrate on in the following.

For general orientation, it is useful to note that if we assume $\text{Re } \Sigma, \text{Im } \Sigma \ll Q$, as is parametrically the case at low energies, then eq. (3.12) can be simplified to

$$\frac{dN_I(x, \mathbf{q})}{d^4x d^3\mathbf{q}} \approx \frac{4n_F(q^0)}{(2\pi)^3 2q^0} \sum_{\alpha=1}^3 \frac{|M_D|_{\alpha I}^2}{Q^2} \text{Tr} \left[\not{Q} a_L \text{Im } \not{\Sigma} a_R \right]. \quad (3.13)$$

Given that the large-time decay of the retarded propagator Π^R is determined by the structure $q^0 + i \text{Im } \Sigma^0$ (cf. eq. (3.11)), and given our conventions in eq. (B.4), we expect the behaviour $\Pi^R(x^0) \sim \int_{q^0} \exp(-iq^0 x^0)/(q^0 + i \text{Im } \Sigma^0) \sim \exp(-x^0 \text{Im } \Sigma^0)$. Therefore $\text{Im } \Sigma^0$ has to be *positive*; in fact, we can define $\text{Im } \Sigma^0 = \Gamma_\nu/2$, where Γ_ν is called the active neutrino damping rate. As eq. (3.13) shows, $\text{Im } \Sigma^0 > 0$ also leads to a positive sterile neutrino production rate.

3.3 Relation of active neutrino and mesonic spectral functions

With the conventions set, we need to determine $i\tilde{\Sigma}_{\alpha\alpha}(\tilde{Q})$. A simple computation to second order in the Fermi interaction (as already mentioned, the first-order contribution vanishes in the absence of chemical potentials) yields

$$i\tilde{\Sigma}_{\alpha\alpha}(\tilde{Q}) = 4G_F^2 \sum_{H=W,Z} \int_{\tilde{R}_b} p_H \tilde{\gamma}_\mu \frac{i\tilde{Q} + i\tilde{R}}{(\tilde{Q} + \tilde{R})^2 + m_{l_H}^2} \tilde{\gamma}_\nu \tilde{C}_{\mu\nu}^H(\tilde{R}), \quad (3.14)$$

where $p_W \equiv 2$, $p_Z \equiv 1/2$ are the “weights” of the charged and neutral current channels; $m_{l_W} \equiv m_{l_\alpha}$ is the mass of the charged lepton of generation α ; $m_{l_Z} \equiv m_{\nu_\alpha} = 0$ is the mass of the MSM active neutrino; $\tilde{R}_b \equiv (\tilde{r}_0, \mathbf{r})$ denotes bosonic Matsubara four-momenta; and we have defined the Euclidean charged and neutral current correlators in accordance with eq. (A.8), *viz.*

$$\tilde{C}_{\mu\nu}^W(\tilde{R}) \equiv \int_{\tilde{x}} e^{i\tilde{R}\cdot(\tilde{x}-\tilde{y})} \left\langle \tilde{H}_\mu^W(\tilde{x}) \tilde{H}_\nu^{W\dagger}(\tilde{y}) \right\rangle, \quad \tilde{C}_{\mu\nu}^Z(\tilde{R}) \equiv \int_{\tilde{x}} e^{i\tilde{R}\cdot(\tilde{x}-\tilde{y})} \left\langle \tilde{H}_\mu^Z(\tilde{x}) \tilde{H}_\nu^Z(\tilde{y}) \right\rangle, \quad (3.15)$$

where $\tilde{x}_\mu \equiv (\tilde{x}^0, \mathbf{x}^i)$, $\tilde{r}_\mu \equiv (\tilde{r}_0, \tilde{r}_i) \equiv (\tilde{r}_0, -r^i)$, $\tilde{x} \cdot \tilde{R} \equiv \tilde{x}_\mu \tilde{r}_\mu = \tilde{x}^0 \tilde{r}_0 - \mathbf{x}^i r^i$, and $\int_{\tilde{x}} \equiv \int_0^\beta d\tilde{x}^0 \int d^3\mathbf{x}$. The tildes in \tilde{C} 's and \tilde{H} 's remind us of the fact that we are for the moment using Euclidean Dirac-matrices in the hadronic currents. We also point out that the Dirac algebra remaining in eq. (3.14) cannot in general be greatly simplified, since the

functions $\tilde{C}_{\mu\nu}^H(\tilde{R})$ can in principle contain both symmetric (e.g. $\tilde{R}_\mu\tilde{R}_\nu$) and antisymmetric ($\epsilon_{\alpha\beta\mu\nu}\tilde{u}_\alpha\tilde{R}_\beta$) structures in $\mu \leftrightarrow \nu$.

It is important to stress now that in eq. (3.14) we have assumed a free form for the lepton/neutrino propagators inside the loop. Naturally, one could also allow for a general structure, such as the one in eq. (3.10), to appear here. To keep the discussion as simple as possible, however, we treat the inner lines to zeroth order in G_F in this paper.

To see how eq. (3.14) can be analysed, let us simplify the notation somewhat. The essential issue is what happens with the Matsubara frequencies, and we hence rewrite the structure as

$$i\tilde{\mathcal{Z}}(\tilde{q}_0, \mathbf{q}) \equiv \int_{\tilde{R}_b} \frac{\tilde{f}_{\mu\nu}(i[\tilde{q}_0 + \tilde{r}_0])}{(\tilde{q}_0 + \tilde{r}_0)^2 + E_1^2} \tilde{C}_{\mu\nu}^H(\tilde{r}_0, \mathbf{r}) \quad \begin{array}{c} \tilde{Q} \\ \circlearrowleft \\ \tilde{R} \\ \circlearrowright \\ \tilde{Q} + \tilde{R} \end{array} \quad , \quad (3.16)$$

where

$$E_1 \equiv \sqrt{(\mathbf{q} + \mathbf{r})^2 + m_{l_H}^2} . \quad (3.17)$$

The drawing in eq. (3.16) illustrates the momentum flow, with the thick line indicating the composite mesonic propagator. The challenge is simply to rewrite eq. (3.16) in a form where the analytic continuation needed for $\text{Re } \Sigma$ and $\text{Im } \Sigma$ can be carried out in a controlled way, without generating any non-converging sums or integrals.

In order to proceed, we first rewrite eq. (3.16) as

$$i\tilde{\mathcal{Z}}(\tilde{q}_0, \mathbf{q}) = \int_{\mathbf{r}} T \sum_{\tilde{r}_{ob}} \sum_{\tilde{s}_{of}} \delta_{\tilde{s}_0 - \tilde{q}_0 - \tilde{r}_0} \frac{\tilde{f}_{\mu\nu}(i\tilde{s}_0)}{\tilde{s}_0^2 + E_1^2} \tilde{C}_{\mu\nu}^H(\tilde{r}_0, \mathbf{r}) , \quad (3.18)$$

where $\int_{\mathbf{r}} \equiv \int d^3\mathbf{r}/(2\pi)^3$, and $\tilde{r}_{ob}, \tilde{s}_{of}$ denote bosonic and fermionic Matsubara frequencies, respectively. Furthermore, we note that the Kronecker δ -function can be expressed as

$$\delta_{\tilde{s}_0 - \tilde{q}_0 - \tilde{r}_0} = T \int_0^\beta d\tau e^{i\tau(\tilde{s}_0 - \tilde{q}_0 - \tilde{r}_0)} . \quad (3.19)$$

Thereby the correlation function becomes

$$i\tilde{\mathcal{Z}}(\tilde{q}_0, \mathbf{q}) = \int_{\mathbf{r}} \int_0^\beta d\tau e^{-i\tau\tilde{q}_0} \left[T \sum_{\tilde{s}_{of}} \frac{\tilde{f}_{\mu\nu}(i\tilde{s}_0)}{\tilde{s}_0^2 + E_1^2} e^{i\tau\tilde{s}_0} \right] \left[T \sum_{\tilde{r}_{ob}} e^{-i\tau\tilde{r}_0} \tilde{C}_{\mu\nu}^H(\tilde{r}_0, \mathbf{r}) \right] . \quad (3.20)$$

The sum inside the first square brackets can be performed according to eq. (B.14). In order to handle the second square brackets, we express $\tilde{C}_{\mu\nu}^H(\tilde{r}_0, \mathbf{r})$ through the spectral representation in eq. (A.14). Inserting this into eq. (3.20) and changing orders of integration, we obtain

$$\begin{aligned} i\tilde{\mathcal{Z}}(\tilde{q}_0, \mathbf{q}) &= \int_{\mathbf{r}} \frac{n_F(E_1)}{2E_1} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \tilde{\rho}_{\mu\nu}^H(\omega, \mathbf{r}) \times \\ &\times \int_0^\beta d\tau e^{-i\tau\tilde{q}_0} \left[\tilde{f}_{\mu\nu}(-E_1) e^{(\beta-\tau)E_1} - \tilde{f}_{\mu\nu}(E_1) e^{\tau E_1} \right] T \sum_{\tilde{r}_{ob}} \frac{e^{-i\tau\tilde{r}_0}}{\omega - i\tilde{r}_0} . \end{aligned} \quad (3.21)$$

As the next step, the sum in eq. (3.21) can be performed. In fact, the result can immediately be obtained by choosing suitable constants c, d in eq. (A.17): for $0 < \tau < \beta$,

$$T \sum_{\tilde{r}_{0b}} \frac{e^{-i\tau\tilde{r}_0}}{\omega - i\tilde{r}_0} = n_B(\omega) e^{(\beta-\tau)\omega}, \quad (3.22)$$

where $n_B(x) \equiv 1/[\exp(\beta x) - 1]$. The integral over τ can then be carried out according to eq. (B.15), leading finally to

$$\begin{aligned} i\tilde{\mathcal{P}}(\tilde{q}_0, \mathbf{q}) &= \int_{\mathbf{r}} \frac{n_F(E_1)}{2E_1} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \tilde{\rho}_{\mu\nu}^H(\omega, \mathbf{r}) n_B(\omega) \times \\ &\times \left[\tilde{f}_{\mu\nu}(-E_1) \frac{e^{\beta(\omega+E_1)} + 1}{i\tilde{q}_0 + \omega + E_1} - \tilde{f}_{\mu\nu}(E_1) \frac{e^{\beta\omega} + e^{\beta E_1}}{i\tilde{q}_0 + \omega - E_1} \right]. \end{aligned} \quad (3.23)$$

It remains to carry out the analytic continuation $\tilde{q}_0 \rightarrow -i[q^0 \pm i0^+]$ and to take the real and imaginary parts. In particular, taking the imaginary part as defined by eq. (3.3) goes with eq. (A.12), given that \tilde{q}_0 only appears in a simple way in eq. (3.23). Carrying then out the integration over ω to remove the δ -functions, and returning simultaneously to Minkowskian Dirac-matrices ($\tilde{\gamma}_\mu \tilde{\gamma}_\mu = \gamma^\mu \gamma_\mu$), whereby the tildes can be removed, we arrive at

$$\begin{aligned} \text{Im } \mathcal{P}(q^0, \mathbf{q}) &= \int_{\mathbf{r}} \frac{\cosh(\beta q^0/2)}{4E_1 \cosh(\beta E_1/2)} \times \\ &\times \left\{ \frac{f^{\mu\nu}(-E_1) \rho_{\mu\nu}^H(-q^0 - E_1, \mathbf{r})}{\sinh[\beta(q^0 + E_1)/2]} - \frac{f^{\mu\nu}(E_1) \rho_{\mu\nu}^H(-q^0 + E_1, \mathbf{r})}{\sinh[\beta(q^0 - E_1)/2]} \right\}. \end{aligned} \quad (3.24)$$

Defining

$$\Delta(p^0, \mathbf{p}, m) \equiv \mathcal{P} + m, \quad (3.25)$$

and reintroducing masses in the numerators to remind us of the fact that the Δ -functions are to be evaluated at the on-shell points (the masses are in any case deleted by the chiral projectors), we can return to the complete notation:

$$\begin{aligned} \text{Im } \mathcal{P}_{\alpha\alpha}(q^0, \mathbf{q}) &= 4G_F^2 \sum_{H=W,Z} p_H \int \frac{d^3\mathbf{r}}{(2\pi)^3} \frac{\cosh(\beta q^0/2)}{4E_1 \cosh(\beta E_1/2)} \times \\ &\times \left[\frac{\gamma^\mu \Delta(-E_1, \mathbf{q} + \mathbf{r}, -m_{l_H}) \gamma^\nu}{\sinh[\beta(q^0 + E_1)/2]} \rho_{\mu\nu}^H(-q^0 - E_1, \mathbf{r}) - (E_1 \rightarrow -E_1) \right]. \end{aligned} \quad (3.26)$$

Here E_1 is from eq. (3.17). Let us note that $\rho_{\mu\nu}^H$ vanishes at zero frequency, so that the poles originating from the sinh-functions in eq. (3.26) are harmless.

To conclude, $\text{Im } \Sigma$ can be expressed as a three-dimensional spatial integral, with certain hyperbolic weights, over the mesonic spectral functions $\rho_{\mu\nu}^H$ related to the charged and neutral currents. On the other hand, $\text{Re } \Sigma$ requires a four-dimensional principal value integration over the same spectral functions, as dictated by eqs. (3.23), (A.12).

3.4 Reduction of mesonic spectral functions

If we make certain assumptions about the quark mass matrix, the mesonic correlators in eq. (3.15), as well as the corresponding spectral functions $\rho_{\mu\nu}^H$ that appear in eq. (3.26), can be reduced to a small set of quantities that have been widely addressed in the literature.

To start with, it is probably a good approximation at temperatures $100 \text{ MeV} \lesssim T \lesssim 400 \text{ MeV}$ to treat the three lightest quarks as degenerate, with a certain mass m_q , while the three heaviest quarks can be assumed infinitely heavy, and ignored. In this limit the theory has an exact $SU_V(3)$ flavour symmetry, which guarantees that we can split the correlators into flavour singlets and non-singlets. Defining T^a to be traceless and Hermitean, and T^0 to be the 3×3 unit matrix, we can then construct the flavour non-singlet and singlet vector and axial currents,

$$\tilde{V}_\mu^a \equiv \bar{\psi} \tilde{\gamma}_\mu T^a \psi, \quad \tilde{V}_\mu^0 \equiv \bar{\psi} \tilde{\gamma}_\mu T^0 \psi, \quad \tilde{A}_\mu^a \equiv \bar{\psi} \tilde{\gamma}_\mu \gamma_5 T^a \psi, \quad \tilde{A}_\mu^0 \equiv \bar{\psi} \tilde{\gamma}_\mu \gamma_5 T^0 \psi. \quad (3.27)$$

In general, the flavour symmetry allows for six correlation functions:

$$\text{Tr}[T^a T^b] \tilde{C}_{\mu\nu}^V(\tilde{R}) \equiv \int_{\tilde{x}} e^{i\tilde{R} \cdot \tilde{x}} \langle \tilde{V}_\mu^a(\tilde{x}) \tilde{V}_\nu^b(0) \rangle, \quad (3.28)$$

$$\text{Tr}[T^a T^b] \tilde{C}_{\mu\nu}^A(\tilde{R}) \equiv \int_{\tilde{x}} e^{i\tilde{R} \cdot \tilde{x}} \langle \tilde{A}_\mu^a(\tilde{x}) \tilde{A}_\nu^b(0) \rangle, \quad (3.29)$$

$$\text{Tr}[T^a T^b] \tilde{C}_{\mu\nu}^{VA}(\tilde{R}) \equiv \int_{\tilde{x}} e^{i\tilde{R} \cdot \tilde{x}} \langle \tilde{V}_\mu^a(\tilde{x}) \tilde{A}_\nu^b(0) + \tilde{A}_\mu^a(\tilde{x}) \tilde{V}_\nu^b(0) \rangle, \quad (3.30)$$

$$\tilde{C}_{\mu\nu}^{V0}(\tilde{R}) \equiv \int_{\tilde{x}} e^{i\tilde{R} \cdot \tilde{x}} \langle \tilde{V}_\mu^0(\tilde{x}) \tilde{V}_\nu^0(0) \rangle, \quad (3.31)$$

$$\tilde{C}_{\mu\nu}^{A0}(\tilde{R}) \equiv \int_{\tilde{x}} e^{i\tilde{R} \cdot \tilde{x}} \langle \tilde{A}_\mu^0(\tilde{x}) \tilde{A}_\nu^0(0) \rangle, \quad (3.32)$$

$$\tilde{C}_{\mu\nu}^{VA0}(\tilde{R}) \equiv \int_{\tilde{x}} e^{i\tilde{R} \cdot \tilde{x}} \langle \tilde{V}_\mu^0(\tilde{x}) \tilde{A}_\nu^0(0) + \tilde{A}_\mu^0(\tilde{x}) \tilde{V}_\nu^0(0) \rangle. \quad (3.33)$$

However, only four among these are non-trivial in the mass-degenerate limit that we are considering: the vector and axial currents have opposite transformation properties in charge conjugation C , which implies that $\tilde{C}_{\mu\nu}^{VA}(\tilde{R})$, $\tilde{C}_{\mu\nu}^{VA0}(\tilde{R})$ vanish in this case.

Now, the correlators that appear in eq. (3.15) can be expressed in terms of the ones just defined. We obtain

$$\tilde{C}_{\mu\nu}^W(\tilde{R}) = \frac{|V_{ud}|^2 + |V_{us}|^2}{4} \left[\tilde{C}_{\mu\nu}^V(\tilde{R}) + \tilde{C}_{\mu\nu}^A(\tilde{R}) \right], \quad (3.34)$$

$$\tilde{C}_{\mu\nu}^Z(\tilde{R}) = \frac{2}{3} \left[(1 - 2x_w)^2 \tilde{C}_{\mu\nu}^V(\tilde{R}) + \tilde{C}_{\mu\nu}^A(\tilde{R}) \right] + \frac{1}{36} \left[\tilde{C}_{\mu\nu}^{V0}(\tilde{R}) + \tilde{C}_{\mu\nu}^{A0}(\tilde{R}) \right], \quad (3.35)$$

where V_{ij} are elements of the CKM matrix. Identical relations hold for the spectral functions.

Under further assumptions, the set of independent correlators can still be reduced. In particular, taking the chiral limit $m_q \rightarrow 0$, the Ward identity following from applying an infinitesimal non-singlet axial transformation on the correlator $\langle \tilde{V}_\mu^a(\tilde{x}) \tilde{A}_\nu^b(0) \rangle$ states that $\tilde{C}_{\mu\nu}^V(\tilde{R}) = \tilde{C}_{\mu\nu}^A(\tilde{R})$. For the singlets this is not true in general, in spite of the fact that

anomalous processes are in some sense suppressed at high temperatures (see, e.g., ref. [30]). So in principle there remain three independent functions to determine, $\tilde{C}_{\mu\nu}^V$, $\tilde{C}_{\mu\nu}^{V0}$, $\tilde{C}_{\mu\nu}^{A0}$.

Now, in the three-flavour theory, the non-singlet vector correlator $\tilde{C}_{\mu\nu}^V$ determines directly the photon spectral function [22], and is thus relevant for computing the photon and dilepton production rates, which are among the prime observables for heavy ion collision experiments. Therefore it has been addressed with a variety of methods in the literature, starting with various perturbative treatments [22] as well as with so-called thermal sum rules [31]. The (resummed [32]) perturbative treatments have reached a great degree of sophistication by now [33], with different strategies applicable in different parts of the phase space. On the other hand, it has also been realised that an analytic continuation of imaginary time correlators defined on the finite τ -interval *can* be carried out in principle [34], which opens up the possibility of lattice QCD determinations. There have indeed been attempts at practical implementations of a certain analytic continuation from numerical data [35], though they are not without problems for the moment [36]. Finally, for $T \ll 150$ MeV, chiral perturbation theory can be systematically applied [37] or at least used as a solid baseline for the computation of $\tilde{C}_{\mu\nu}^V$ (and thus of $\rho_{\mu\nu}^V$) [38].

The vector singlet $\tilde{C}_{\mu\nu}^{V0}$, on the other hand, can be associated with the baryon number current, whose susceptibility χ (which is an integral over the spectral function, $\chi = \int_{-\infty}^{\infty} d\omega \rho_{00}^{V0}(\omega, \mathbf{0})/\pi\omega$) is argued to be relevant for so-called event-to-event fluctuations in heavy ion collision experiments [39]. In resummed perturbative treatments [40] the difference between the vector singlet and non-singlet susceptibilities (this difference is often called the “off-diagonal quark number susceptibility”) is however very small, being suppressed by $\alpha_s^3 \ln(1/\alpha_s)$ [41], so that at high enough temperatures $\tilde{C}_{\mu\nu}^{V0}$ can well be approximated (up to an overall factor) by $\tilde{C}_{\mu\nu}^V$. At lower temperatures close to $T \simeq 150$ MeV, on the other hand, a lattice determination would again be needed; unfortunately, the difference between $\tilde{C}_{\mu\nu}^{V0}$ and $\tilde{C}_{\mu\nu}^V$ is given by a disconnected quark-line contraction which is technically rather difficult to measure accurately [42]. We should of course stress that the susceptibility alone contains much less information than the complete function $\tilde{C}_{\mu\nu}^{V0}$, or its analytic continuation $\rho_{\mu\nu}^{V0}$. The general high-temperature structure of the latter has been analysed in ref. [43]. Finally, at very low temperatures, chiral perturbation theory predicts that $\tilde{C}_{\mu\nu}^{V0}$ is strongly suppressed with respect to $\tilde{C}_{\mu\nu}^V$.

Much the same comments can be made for the axial singlet current, $\tilde{C}_{\mu\nu}^{A0}$. At high enough temperatures, where resummed QCD perturbation theory is applicable, it agrees up to a certain order in the resummed perturbative expansion with $\tilde{C}_{\mu\nu}^{V0}$. At lower temperatures, the difference between $\tilde{C}_{\mu\nu}^{A0}$ and $\tilde{C}_{\mu\nu}^{V0}$ becomes significant. This difference is of course quite interesting in its own right, being related to the η' -meson and the chiral anomaly. However, analytic and numerical treatments are demanding in this range. At very low temperatures, chiral perturbation theory indicates that $\tilde{C}_{\mu\nu}^{A0}$ remains more significant than $\tilde{C}_{\mu\nu}^{V0}$.

3.5 Perturbative limit

Returning finally to the simplest possible logic, we wish to inspect the mesonic spectral

function $\rho_{\mu\nu}^H$ in naive perturbation theory. The purpose is to show that in this case, eq. (3.26) leads to the familiar structure of Boltzmann equations. (The procedure we follow is analogous to the one first worked out for simpler cases in ref. [44], and also used in previous active neutrino damping rate computations [45].)

In order to reach this goal, we write generic hadronic currents (eqs. (3.6), (3.7)) in the form $\tilde{H}_\mu(\tilde{x}) = \bar{q}_2(\tilde{x})\tilde{\gamma}_\mu\Gamma q_3(\tilde{x})$, where q_2, q_3 denote quark fields and Γ is some Dirac matrix structure. Carrying then out the contractions in the correlators of eq. (3.15), we obtain

$$\tilde{C}_{\mu\nu}^H(\tilde{R}) = -N_c \int_{\tilde{T}_f} \text{Tr} \left\{ \frac{-i(\tilde{\mathcal{T}} + \tilde{\mathcal{R}}) + m_2}{(\tilde{t}_0 + \tilde{r}_0)^2 + E_2^2} \tilde{\gamma}_\mu \Gamma \frac{-i\tilde{\mathcal{T}} + m_3}{\tilde{t}_0^2 + E_3^2} \tilde{\gamma}_\nu \Gamma \right\}, \quad (3.36)$$

where $N_c = 3$ is the number of colours, and

$$E_2 \equiv \sqrt{(\mathbf{t} + \mathbf{r})^2 + m_2^2}, \quad E_3 \equiv \sqrt{\mathbf{t}^2 + m_3^2}. \quad (3.37)$$

The important issue is again what happens with the Matsubara frequencies. Following the steps in section 3.3, and denoting the complete numerator of eq. (3.36) by a function $\tilde{g}_{\mu\nu}$, we obtain

$$\begin{aligned} T \sum_{\tilde{t}_{\text{of}}} \frac{\tilde{g}_{\mu\nu}(i\tilde{t}_0 + i\tilde{r}_0, i\tilde{t}_0)}{[(\tilde{t}_0 + \tilde{r}_0)^2 + E_2^2][\tilde{t}_0^2 + E_3^2]} \\ &= T \sum_{\tilde{t}_{\text{of}}, \tilde{u}_{\text{of}}} \delta_{\tilde{u}_0 - \tilde{t}_0 - \tilde{r}_0} \frac{\tilde{g}_{\mu\nu}(i\tilde{u}_0, i\tilde{t}_0)}{[\tilde{u}_0^2 + E_2^2][\tilde{t}_0^2 + E_3^2]} \\ &= \int_0^\beta d\tau e^{-i\tau\tilde{r}_0} T \sum_{\tilde{u}_{\text{of}}} e^{i\tilde{u}_0\tau} T \sum_{\tilde{t}_{\text{of}}} e^{-i\tilde{t}_0\tau} \frac{\tilde{g}_{\mu\nu}(i\tilde{u}_0, i\tilde{t}_0)}{[\tilde{u}_0^2 + E_2^2][\tilde{t}_0^2 + E_3^2]} \\ &= \frac{n_F(E_2)n_F(E_3)}{4E_2E_3} \int_0^\beta d\tau e^{-i\tau\tilde{r}_0} \left[+\tilde{g}_{\mu\nu}(-E_2, +E_3)e^{(\beta-\tau)(E_2+E_3)} - \right. \\ &\quad -\tilde{g}_{\mu\nu}(-E_2, -E_3)e^{(\beta-\tau)E_2+\tau E_3} - \\ &\quad -\tilde{g}_{\mu\nu}(+E_2, +E_3)e^{(\beta-\tau)E_3+\tau E_2} + \\ &\quad \left. +\tilde{g}_{\mu\nu}(+E_2, -E_3)e^{\tau(E_2+E_3)} \right], \quad (3.38) \end{aligned}$$

where we made use of eq. (B.14). The integral remaining can be carried out by using eq. (A.18) (recalling that \tilde{r}_0 is bosonic), and the spectral function follows then by picking up the discontinuity across the real axis, according to eq. (A.12):

$$\begin{aligned} \tilde{\rho}_{\mu\nu}^H(R) = \int_{\mathbf{t}} \frac{-\pi}{4E_2E_3} \left[+\delta(r^0 + E_2 + E_3)\tilde{g}_{\mu\nu}(-E_2, +E_3)(1 - n_{F2} - n_{F3}) + \right. \\ +\delta(r^0 + E_2 - E_3)\tilde{g}_{\mu\nu}(-E_2, -E_3)(n_{F2} - n_{F3}) + \\ +\delta(r^0 - E_2 + E_3)\tilde{g}_{\mu\nu}(+E_2, +E_3)(n_{F3} - n_{F2}) + \\ \left. +\delta(r^0 - E_2 - E_3)\tilde{g}_{\mu\nu}(+E_2, -E_3)(n_{F2} + n_{F3} - 1) \right], \quad (3.39) \end{aligned}$$

where we have denoted $n_{Fi} \equiv n_F(E_i)$.

The expression in eq. (3.39) can now be inserted into eq. (3.26). There are eight different terms in total. The hyperbolic functions in eq. (3.26) can be rewritten in terms of n_F 's and n_B 's, and making use of relations such as

$$\delta(-q^0 + E_1 + E_2 + E_3)n_B(q^0 - E_1)(1 - n_{F2} - n_{F3}) = \delta(-q^0 + E_1 + E_2 + E_3)n_{F2}n_{F3}, \quad (3.40)$$

they can be reorganised in a simpler form. In view of eq. (2.21), we also choose to factor out $n_F^{-1}(q^0)$ from all the terms. After a number of trivial if tedious manipulations, we finally arrive at

$$\begin{aligned} \text{Im } \mathcal{Y}(Q) = & 2N_c G_F^2 n_F^{-1}(q^0) \sum_{H=W,Z} p_H \int \frac{d^3 \mathbf{p}_1}{(2\pi)^3 2E_1} \int \frac{d^3 \mathbf{p}_2}{(2\pi)^3 2E_2} \int \frac{d^3 \mathbf{p}_3}{(2\pi)^3 2E_3} \times \\ & \times \left\{ (2\pi)^4 \delta^{(4)}(P_1 + P_2 + P_3 - Q) n_{F1} n_{F2} n_{F3} \mathcal{A}(-m_{l_H}, m_2, -m_3) + \begin{array}{c} 1 \\ \nearrow \\ 2 \rightarrow \bullet \leftarrow Q \\ \searrow \\ 3 \end{array} \right. \\ & + (2\pi)^4 \delta^{(4)}(P_2 + P_3 - P_1 - Q) n_{F2} n_{F3} (1 - n_{F1}) \mathcal{A}(m_{l_H}, m_2, -m_3) + \begin{array}{c} 2 \\ \nearrow \\ 3 \rightarrow \bullet \leftarrow Q \\ \searrow \\ 1 \end{array} \\ & + (2\pi)^4 \delta^{(4)}(P_1 + P_3 - P_2 - Q) n_{F1} n_{F3} (1 - n_{F2}) \mathcal{A}(-m_{l_H}, -m_2, -m_3) + \begin{array}{c} 1 \\ \nearrow \\ 3 \rightarrow \bullet \leftarrow Q \\ \searrow \\ 2 \end{array} \\ & + (2\pi)^4 \delta^{(4)}(P_1 + P_2 - P_3 - Q) n_{F1} n_{F2} (1 - n_{F3}) \mathcal{A}(-m_{l_H}, m_2, m_3) + \begin{array}{c} 1 \\ \nearrow \\ 2 \rightarrow \bullet \leftarrow Q \\ \searrow \\ 3 \end{array} \\ & + (2\pi)^4 \delta^{(4)}(P_1 - P_2 - P_3 - Q) n_{F1} (1 - n_{F2})(1 - n_{F3}) \mathcal{A}(-m_{l_H}, -m_2, m_3) + \begin{array}{c} 1 \\ \nearrow \\ 2 \rightarrow \bullet \leftarrow Q \\ \searrow \\ 3 \end{array} \\ & + (2\pi)^4 \delta^{(4)}(P_2 - P_1 - P_3 - Q) n_{F2} (1 - n_{F1})(1 - n_{F3}) \mathcal{A}(m_{l_H}, m_2, m_3) + \begin{array}{c} 2 \\ \nearrow \\ 3 \rightarrow \bullet \leftarrow Q \\ \searrow \\ 1 \end{array} \\ & + (2\pi)^4 \delta^{(4)}(P_3 - P_1 - P_2 - Q) n_{F3} (1 - n_{F1})(1 - n_{F2}) \mathcal{A}(m_{l_H}, -m_2, -m_3) + \begin{array}{c} 3 \\ \nearrow \\ 2 \rightarrow \bullet \leftarrow Q \\ \searrow \\ 1 \end{array} \\ & + (2\pi)^4 \delta^{(4)}(-P_1 - P_2 - P_3 - Q) (1 - n_{F1})(1 - n_{F2})(1 - n_{F3}) \mathcal{A}(m_{l_H}, -m_2, m_3) \left. \right\}, \begin{array}{c} 1 \\ \nearrow \\ 2 \rightarrow \bullet \leftarrow Q \\ \searrow \\ 3 \end{array} \quad (3.41) \end{aligned}$$

where

$$\mathcal{A}(m_{l_H}, m_2, m_3) \equiv \gamma^\mu (\not{P}_1 + m_{l_H}) \gamma^\nu \text{Tr} \left[(\not{P}_2 + m_2) \gamma_\mu \Gamma (\not{P}_3 + m_3) \gamma_\nu \Gamma \right]. \quad (3.42)$$


Here $P_i \equiv (E_i, \mathbf{p}_i)$ are on-shell four-momenta, and the energies have been transformed from eqs. (3.17), (3.37) into

$$E_1 \equiv \sqrt{\mathbf{p}_1^2 + m_{l_H}^2}, \quad E_2 \equiv \sqrt{\mathbf{p}_2^2 + m_2^2}, \quad E_3 \equiv \sqrt{\mathbf{p}_3^2 + m_3^2}. \quad (3.43)$$

The graphs in eq. (3.41) illustrate the various processes, with time and momenta assumed to run from left to right, and arrows indicating particles / antiparticles in the usual way. The general structure is clearly what we expect from Boltzmann equations, however we have arrived at it without any model assumptions, evaluations of scattering matrix elements, or spin averages. It is easy to check (by making use of the properties of the n_F 's as well as the fact that the first argument of $\mathcal{A}(m_1, m_2, m_3)$ can be dropped as it will in any case be

projected out by a_L, a_R , and that $\mathcal{A}(0, -m_2, -m_3) = \mathcal{A}(0, m_2, m_3)$ for $\Gamma = a + b\gamma_5$) that the expression in eq. (3.41) is symmetric in $Q \rightarrow -Q$, as it should be.

For illustration it is useful to take one more step, and insert this result into the simplified form in eq. (3.13). Let us consider the contribution from the third term in eq. (3.41), for instance, and choose the charged current, i.e. $H \equiv W$. Then $\Gamma = a_L$, and we obtain

$$\begin{aligned} \frac{dN_I(x, \mathbf{q})}{d^4x d^3\mathbf{q}} &= \frac{16N_c G_F^2}{(2\pi)^3 2q^0} \sum_{\alpha=1}^3 \frac{|M_D|_{\alpha I}^2}{M_I^2} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \int \frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \times \\ &\times (2\pi)^4 \delta^{(4)}(P_1 + P_3 - P_2 - Q) n_F(u \cdot P_1) n_F(u \cdot P_3) [1 - n_F(u \cdot P_2)] \times \\ &\times \text{Tr} \left[Q a_L \gamma^\mu (\not{P}_1 - m_{\nu_\alpha}) \gamma^\nu a_R \right] \text{Tr} \left[(\not{P}_2 - m_2) \gamma_\mu a_L (\not{P}_3 - m_3) \gamma_\nu a_L \right], \end{aligned} \quad (3.44)$$


where $m_{\nu_\alpha} = 0$ has only been kept to formally indicate the anti-particle direction of the line, and we have written everything in a Lorentz-covariant form. All explicit masses drop out because of the chiral projectors. The Dirac algebra is elementary,

$$\text{Tr} \left[Q \gamma^\mu \not{P}_1 \gamma^\nu a_R \right] \text{Tr} \left[\not{P}_2 \gamma_\mu \not{P}_3 \gamma_\nu a_L \right] = 16 Q \cdot P_3 P_1 \cdot P_2. \quad (3.45)$$

This expression is positive, and carrying out the phase-space integral in eq. (3.44), one obtains a finite function of $|\mathbf{q}|$. Let us remark, though, that while the integration over the energy-conservation constraint is simple in the center-of-mass frame, the plasma four-velocity u becomes non-trivial if this frame is chosen; in general, therefore, the phase-space integrals that appear in eq. (3.44) are technically non-trivial.

It is appropriate to end now by pointing out that eqs. (3.44), (3.45) contain significant differences with respect to the spin-averaged matrix elements squared that appear in active neutrino scattering cross-sections [45], and have sometimes also been inserted into the Boltzmann equations for sterile neutrino production. In particular, the left-most trace in eq. (3.45) contains the projector a_R , rather than a_L ; this is because eq. (2.21) contains the active neutrino propagator, rather than the self-energy. As a consequence, eq. (3.45) does not lead to a purely s -channel momentum structure ($16 Q \cdot P_2 P_1 \cdot P_3$) like the active neutrino scattering cross sections [45]. Whether this special example has any practical significance is not clear at this stage, but it illustrates the advantages of our framework where the correct structures are produced automatically from thermal field theory, rather than having to be inserted by hand.

4. Conclusions and outlook

While the sterile neutrino production rate has previously been investigated in the literature in some detail, the hadronic contributions to it have never been analysed properly. These contributions involve strongly interacting dynamics at temperatures of the order of the QCD scale, where neither perturbation theory nor the dilute hadronic gas approximation are valid.

To confront this situation, we have derived a general relation that expresses the sterile neutrino production rate in terms of the active neutrino spectral function, computable by

using equilibrium thermal field theory within the Minimal Standard Model (eq. (2.21)). The active neutrino spectral function can in turn be expressed in terms of the real and imaginary parts of the active neutrino self-energy (eq. (3.12)). These equations show that hadronic contributions may play a role in three different ways:

- (1) Most importantly, the hadronic degrees of freedom contribute at leading order to the imaginary part of the active neutrino self-energy, $\text{Im} \Sigma$. Therefore, we have expressed the hadronic contribution to $\text{Im} \Sigma$ as a certain convolution of the spectral functions related to hadronic current-current correlation functions (eq. (3.26)). The latter can in turn be expressed in terms of standard vector and axial-vector correlators (eqs. (3.34), (3.35)) that can be studied with a number of different theoretical methods, and are also partly related to experimental observables addressed in the heavy ion program.
- (2) The parametrically dominant contribution to the real part of the active neutrino self-energy, $\text{Re} \Sigma$, arises from 1-loop graphs within the electroweak theory, and does not contain hadronic effects. On the other hand, subdominant contributions, formally suppressed by α_w , do contain hadronic effects. As shown by eq. (3.23), these contributions can be expressed in terms of a certain weighted integral over the same hadronic spectral functions that appear in the imaginary part.
- (3) Finally, the sterile neutrino production rate equation contains a time derivative d/dt , the Hubble rate H , and the temperature T (cf. eq. (2.22)). In cosmology, all of these are related via the Einstein equations. The relation is again sensitive to hadronic effects, via the equation-of-state of the primordial plasma. The current status and phenomenological fits for all the thermodynamic functions that appear in the time-temperature relation can be found in ref. [21, section IV].

To summarise, our formulae should allow to estimate for the first time the systematic hadronic uncertainties that exist in computations of sterile neutrino production through active-sterile transitions. A numerical evaluation of these effects is in progress.

Apart from these non-perturbative aspects, we have also demonstrated that our general formulae allow to derive, without further assumptions, the appropriate Boltzmann equations that apply in the naive weak-coupling limit (eq. (3.41)). It may in fact be useful to repeat our computations for the leptonic contributions as well, since a first-principles derivation frees us from the need to argue about spin averages or symmetry factors, and produces automatically the correct Dirac and chiral structures for the Boltzmann equations.

Acknowledgments

The work of T.A. was supported in part by the grants-in-aid from the Ministry of Education, Science, Sports, and Culture of Japan, Nos. 16081202 and 17340062, and that of M.S. by the Swiss National Science Foundation.

A. Basic relations for bosons

We list in this Appendix some common definitions and relations that apply to two-point correlation functions built out of bosonic operators; for more details see, e.g., refs. [24, 25].

We denote Minkowskian space-time coordinates by $x = (t, x^i)$ and momenta by $Q = (q^0, q^i)$, while their Euclidean counterparts are denoted by $\tilde{x} = (\tau, x^i)$, $\tilde{Q} = (\tilde{q}_0, q_i)$. Wick rotation is carried out by $\tau \leftrightarrow it$, $\tilde{q}_0 \leftrightarrow -iq^0$. Arguments of operators denote implicitly whether we are in Minkowskian or Euclidean space-time. In particular, Heisenberg-operators are defined as

$$\hat{O}(t, \mathbf{x}) \equiv e^{i\hat{H}t} \hat{O}(0, \mathbf{x}) e^{-i\hat{H}t}, \quad \hat{O}(\tau, \mathbf{x}) \equiv e^{\hat{H}\tau} \hat{O}(0, \mathbf{x}) e^{-\hat{H}\tau}. \quad (\text{A.1})$$

The thermal ensemble is defined by the density matrix $\hat{\rho} = Z^{-1} \exp(-\beta\hat{H})$.

We denote the operators which appear in the two-point functions by $\hat{\phi}_\alpha(x)$, $\hat{\phi}_\beta^\dagger(x)$. They could be elementary field operators, but they could also be composite operators consisting of a product of elementary field operators.

We can now define various classes of correlation functions. The “physical” correlators are defined as

$$\Pi_{\alpha\beta}^>(Q) \equiv \int dt d^3\mathbf{x} e^{iQ \cdot x} \langle \hat{\phi}_\alpha(x) \hat{\phi}_\beta^\dagger(0) \rangle, \quad (\text{A.2})$$

$$\Pi_{\alpha\beta}^<(Q) \equiv \int dt d^3\mathbf{x} e^{iQ \cdot x} \langle \hat{\phi}_\beta^\dagger(0) \hat{\phi}_\alpha(x) \rangle, \quad (\text{A.3})$$

$$\rho_{\alpha\beta}(Q) \equiv \int dt d^3\mathbf{x} e^{iQ \cdot x} \left\langle \frac{1}{2} \left[\hat{\phi}_\alpha(x), \hat{\phi}_\beta^\dagger(0) \right] \right\rangle, \quad (\text{A.4})$$

where $\rho_{\alpha\beta}$ is called the spectral function, while the “retarded”/“advanced” correlators can be defined as

$$\Pi_{\alpha\beta}^R(Q) \equiv i \int dt d^3\mathbf{x} e^{iQ \cdot x} \langle [\hat{\phi}_\alpha(x), \hat{\phi}_\beta^\dagger(0)] \theta(t) \rangle, \quad (\text{A.5})$$

$$\Pi_{\alpha\beta}^A(Q) \equiv i \int dt d^3\mathbf{x} e^{iQ \cdot x} \langle -[\hat{\phi}_\alpha(x), \hat{\phi}_\beta^\dagger(0)] \theta(-t) \rangle. \quad (\text{A.6})$$

On the other hand, from the computational point of view one is often faced with “time-ordered” correlation functions,

$$\Pi_{\alpha\beta}^T(Q) \equiv \int dt d^3\mathbf{x} e^{iQ \cdot x} \langle \hat{\phi}_\alpha(x) \hat{\phi}_\beta^\dagger(0) \theta(t) + \hat{\phi}_\beta^\dagger(0) \hat{\phi}_\alpha(x) \theta(-t) \rangle, \quad (\text{A.7})$$

which appear in time-dependent perturbation theory, or with the “Euclidean” correlator

$$\Pi_{\alpha\beta}^E(\tilde{Q}) \equiv \int_0^\beta d\tau \int d^3\mathbf{x} e^{i\tilde{Q} \cdot \tilde{x}} \langle \hat{\phi}_\alpha(\tilde{x}) \hat{\phi}_\beta^\dagger(0) \rangle, \quad (\text{A.8})$$

which appears in non-perturbative formulations. Note that the Euclidean correlator is also time-ordered by definition, and can be computed with standard imaginary-time functional integrals in the Matsubara formalism.

Now, all of the correlation functions defined can be related to each other. In particular, all correlators can be expressed in terms of the spectral function, which in turn can be determined as a certain analytic continuation of the Euclidean correlator. In order to do this, we may first insert sets of energy eigenstates, to obtain the Fourier-space version of the so-called Kubo-Martin-Schwinger (KMS) relation: $\Pi_{\alpha\beta}^{\leq}(Q) = e^{-\beta q^0} \Pi_{\alpha\beta}^{\geq}(Q)$. Then $\rho_{\alpha\beta}(Q) = [\Pi_{\alpha\beta}^{\geq}(Q) - \Pi_{\alpha\beta}^{\leq}(Q)]/2$ and, conversely,

$$\Pi_{\alpha\beta}^{\geq}(Q) = 2[1 + n_B(q^0)]\rho_{\alpha\beta}(Q), \quad \Pi_{\alpha\beta}^{\leq}(Q) = 2n_B(q^0)\rho_{\alpha\beta}(Q), \quad (\text{A.9})$$

where $n_B(x) \equiv 1/[\exp(\beta x) - 1]$. Inserting the representation

$$\theta(t) = i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i0^+} \quad (\text{A.10})$$

into the definitions of Π^R , Π^A , we obtain

$$\Pi_{\alpha\beta}^R(Q) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho_{\alpha\beta}(\omega, \mathbf{q})}{\omega - q^0 - i0^+}, \quad \Pi_{\alpha\beta}^A(Q) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho_{\alpha\beta}(\omega, \mathbf{q})}{\omega - q^0 + i0^+}. \quad (\text{A.11})$$

Doing the same with Π^T and making use of

$$\frac{1}{\Delta \pm i0^+} = P\left(\frac{1}{\Delta}\right) \mp i\pi\delta(\Delta), \quad (\text{A.12})$$

produces

$$\Pi_{\alpha\beta}^T(Q) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{i\rho_{\alpha\beta}(\omega, \mathbf{q})}{q^0 - \omega + i0^+} + 2\rho_{\alpha\beta}(q^0, \mathbf{q})n_B(q^0). \quad (\text{A.13})$$

Finally, writing the argument inside the τ -integration in eq. (A.8) as a Wick rotation of the integrand in eq. (A.2), which in turn is expressed as an inverse Fourier transform of $\Pi^{\geq}(Q)$, for which eq. (A.9) is inserted, and changing orders of integration, we get

$$\Pi_{\alpha\beta}^E(\tilde{Q}) = \int_0^{\beta} d\tau e^{i\tilde{q}_0\tau} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-\omega\tau} \Pi_{\alpha\beta}^{\geq}(\omega, \mathbf{q}) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho_{\alpha\beta}(\omega, \mathbf{q})}{\omega - i\tilde{q}_0}. \quad (\text{A.14})$$

This relation can formally be inverted by making use of eq. (A.12),

$$\rho_{\alpha\beta}(q^0, \mathbf{q}) = \frac{1}{2i} \text{Disc} \Pi_{\alpha\beta}^E(\tilde{q}_0 \rightarrow -iq^0, \mathbf{q}), \quad (\text{A.15})$$

where the operation Disc is defined in eq. (3.3).

We also recall that bosonic Matsubara sums can be carried out through

$$T \sum_{\omega_b} \frac{i\omega_b c + d}{\omega_b^2 + E^2} e^{i\omega_b \tau} \equiv (c\partial_{\tau} + d) T \sum_{\omega_b} \frac{e^{i\omega_b \tau}}{\omega_b^2 + E^2} \quad (\text{A.16})$$

$$= \frac{n_B(E)}{2E} \left[(-cE + d)e^{(\beta-\tau)E} + (cE + d)e^{\tau E} \right], \quad (\text{A.17})$$

where $\omega_b = 2\pi Tn$, with n an integer, and we assumed $0 < \tau < \beta$; and that a typical integration yields

$$\int_0^{\beta} d\tau e^{-\tau(i\omega_b + \Delta)} = \frac{1 - e^{-\beta\Delta}}{i\omega_b + \Delta}. \quad (\text{A.18})$$

B. Basic relations for fermions

We list in this Appendix some common definitions and relations that apply to two-point correlation functions built out of fermionic operators; for more details see, e.g., Refs. [24, 25].

We denote the operators which appear in the two-point functions by $\hat{\nu}_\alpha(x)$, $\hat{\nu}_\beta(x)$. They could be elementary field operators, in which case the indices α, β label Dirac and/or flavour components, but they could also be composite operators consisting of a product of elementary field operators.

Like in the bosonic case, we can define various classes of correlation functions. The “physical” correlators are now set up as

$$\Pi_{\alpha\beta}^>(Q) \equiv \int dt d^3\mathbf{x} e^{iQ\cdot x} \langle \hat{\nu}_\alpha(x) \hat{\nu}_\beta(0) \rangle, \quad (\text{B.1})$$

$$\Pi_{\alpha\beta}^<(Q) \equiv \int dt d^3\mathbf{x} e^{iQ\cdot x} \langle -\hat{\nu}_\beta(0) \hat{\nu}_\alpha(x) \rangle, \quad (\text{B.2})$$

$$\rho_{\alpha\beta}(Q) \equiv \int dt d^3\mathbf{x} e^{iQ\cdot x} \left\langle \frac{1}{2} \left\{ \hat{\nu}_\alpha(x), \hat{\nu}_\beta(0) \right\} \right\rangle, \quad (\text{B.3})$$

where $\rho_{\alpha\beta}$ is the spectral function, while retarded and advanced correlators can be defined as

$$\Pi_{\alpha\beta}^R(Q) \equiv i \int dt d^3\mathbf{x} e^{iQ\cdot x} \left\langle \left\{ \hat{\nu}_\alpha(x), \hat{\nu}_\beta(0) \right\} \theta(t) \right\rangle, \quad (\text{B.4})$$

$$\Pi_{\alpha\beta}^A(Q) \equiv i \int dt d^3\mathbf{x} e^{iQ\cdot x} \left\langle -\left\{ \hat{\nu}_\alpha(x), \hat{\nu}_\beta(0) \right\} \theta(-t) \right\rangle. \quad (\text{B.5})$$

On the other hand, the time-ordered correlation function reads

$$\Pi_{\alpha\beta}^T(Q) \equiv \int dt d^3\mathbf{x} e^{iQ\cdot x} \left\langle \hat{\nu}_\alpha(x) \hat{\nu}_\beta(0) \theta(t) - \hat{\nu}_\beta(0) \hat{\nu}_\alpha(x) \theta(-t) \right\rangle, \quad (\text{B.6})$$

while the Euclidean correlator is

$$\Pi_{\alpha\beta}^E(\tilde{Q}) \equiv \int_0^\beta d\tau \int d^3\mathbf{x} e^{i\tilde{Q}\cdot\tilde{x}} \left\langle \hat{\nu}_\alpha(\tilde{x}) \hat{\nu}_\beta(0) \right\rangle. \quad (\text{B.7})$$

Note again that the Euclidean correlator is time-ordered by definition, and can be computed with standard imaginary-time functional integrals in the Matsubara formalism.

Like in the bosonic case, all of the correlation functions defined can be expressed in terms of the spectral function, which in turn can be determined as a certain analytic continuation of the Euclidean correlator. First, inserting sets of energy eigenstates, we obtain the KMS-relation in Fourier-space, $\Pi_{\alpha\beta}^<(Q) = -e^{-\beta q^0} \Pi_{\alpha\beta}^>(Q)$. Then $\rho_{\alpha\beta}(Q) = [\Pi_{\alpha\beta}^>(Q) - \Pi_{\alpha\beta}^<(Q)]/2$ and, conversely,

$$\Pi_{\alpha\beta}^>(Q) = 2[1 - n_F(q^0)]\rho_{\alpha\beta}(Q), \quad \Pi_{\alpha\beta}^<(Q) = -2n_F(q^0)\rho_{\alpha\beta}(Q), \quad (\text{B.8})$$

where $n_F(x) \equiv 1/[\exp(\beta x) + 1]$. Inserting the representation of eq. (A.10) into the definitions of Π^R , Π^A produces

$$\Pi_{\alpha\beta}^R(Q) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho_{\alpha\beta}(\omega, \mathbf{q})}{\omega - q^0 - i0^+}, \quad \Pi_{\alpha\beta}^A(Q) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho_{\alpha\beta}(\omega, \mathbf{q})}{\omega - q^0 + i0^+}. \quad (\text{B.9})$$

Proceeding similarly with Π^T and making use of eq. (A.12), we obtain

$$\Pi_{\alpha\beta}^T(Q) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{i\rho_{\alpha\beta}(\omega, \mathbf{q})}{q^0 - \omega + i0^+} - 2\rho_{\alpha\beta}(q^0, \mathbf{q})n_{\text{F}}(q^0). \quad (\text{B.10})$$

Finally, writing the argument inside the τ -integration in eq. (B.7) as a Wick rotation of the inverse Fourier transform of the left-hand side of eq. (B.1), inserting eq. (B.8), and changing orders of integration, we get

$$\Pi_{\alpha\beta}^E(\tilde{Q}) = \int_0^\beta d\tau e^{i\tilde{q}_0\tau} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-\omega\tau} \Pi_{\alpha\beta}^>(\omega, \mathbf{q}) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho_{\alpha\beta}(\omega, \mathbf{q})}{\omega - i\tilde{q}_0}. \quad (\text{B.11})$$

Like in the bosonic case, this relation can be inverted by making use of eq. (A.12),

$$\rho(q^0, \mathbf{q}) = \frac{1}{2i} \text{Disc} \Pi^E(\tilde{q}_0 \rightarrow -iq^0, \mathbf{q}). \quad (\text{B.12})$$

We also recall that fermionic Matsubara sums can be carried out through

$$T \sum_{\omega_{\text{f}}} \frac{i\omega_{\text{f}}c + d}{\omega_{\text{f}}^2 + E^2} e^{i\omega_{\text{f}}\tau} \equiv (c\partial_\tau + d)T \sum_{\omega_{\text{f}}} \frac{e^{i\omega_{\text{f}}\tau}}{\omega_{\text{f}}^2 + E^2} \quad (\text{B.13})$$

$$= \frac{n_{\text{F}}(E)}{2E} \left[(-cE + d)e^{(\beta-\tau)E} - (cE + d)e^{\tau E} \right], \quad (\text{B.14})$$

where $\omega_{\text{f}} = 2\pi T(n + \frac{1}{2})$, with n an integer, and we assumed $0 < \tau < \beta$; and that a typical integration yields

$$\int_0^\beta d\tau e^{-\tau(i\omega_{\text{f}} + \Delta)} = \frac{1 + e^{-\beta\Delta}}{i\omega_{\text{f}} + \Delta}. \quad (\text{B.15})$$

C. An alternative derivation of eq. (2.21)

We present in this Appendix an alternative derivation (following ref. [25], for example) of eq. (2.21), which is technically somewhat simpler than the one in the main text, but with the price of containing a few heuristic steps. The end result is nevertheless identical.

The starting point is the interaction Hamiltonian in the phase with broken electroweak symmetry, eq. (2.13). Consider now an initial state $|\mathcal{I}\rangle = |i\rangle \otimes |0\rangle$ and a final state $|\mathcal{F}\rangle = |f\rangle \otimes |I; \mathbf{q}, s\rangle$, where the right subspace contains the sterile neutrinos, and

$$|I; \mathbf{q}, s\rangle \equiv \hat{a}_{I; \mathbf{q}, s}^\dagger |0\rangle, \quad \langle I; \mathbf{q}, s| = \langle 0| \hat{a}_{I; \mathbf{q}, s}. \quad (\text{C.1})$$

The transition matrix element can immediately be written down,

$$T_{\mathcal{F}\mathcal{I}} = \langle \mathcal{F}| \int dt \hat{H}_I(t) |\mathcal{I}\rangle = \int dt d^3\mathbf{x} \frac{e^{iQ \cdot x}}{\sqrt{(2\pi)^3 2q^0}} \langle f| \hat{J}_{I; \mathbf{q}, s}(x) |i\rangle, \quad (\text{C.2})$$

where we inserted eq. (2.16), and $\hat{J}_{I; \mathbf{q}, s}$ is from eq. (2.17). The production rate can then be obtained by summing over all initial states, with their proper Boltzmann weights, and

over all allowed final states:

$$\frac{dN_I(x, \mathbf{q})}{d^4x d^3\mathbf{q}} = \lim_{V, \Delta t \rightarrow \infty} \frac{1}{V \Delta t} \sum_{s=\pm 1} \sum_{f,i} \frac{e^{-\beta E_i}}{Z} |T_{\mathcal{F}\mathcal{I}}|^2 \quad (\text{C.3})$$

$$= \frac{1}{(2\pi)^3 2q^0} \int dt d^3\mathbf{x} e^{iQ \cdot x} \sum_{s=\pm 1} \left\langle \hat{J}_{I;\mathbf{q},s}^\dagger(x) \hat{J}_{I;\mathbf{q},s}(0) \right\rangle, \quad (\text{C.4})$$

where V is the volume, Δt is the time interval, Z the partition function, we defined a thermal average by $\langle \dots \rangle \equiv Z^{-1} \text{Tr}[\exp(-\beta \hat{H}_{\text{MSM}})(\dots)]$, and made use of translational invariance.

It remains to repeat steps (ii), (iii) in the paragraph following eq. (2.17). We thus arrive directly at eq. (2.20).

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